correct matches are significantly more for the HOS case (76 more data points are correctly classified using HOS as compared to the square distortion measure). It must, however, be noted that the HOS-based HDA algorithm tends to misclassify some far away points in the first cluster.

V. CONCLUSION

In this brief, we have demonstrated the advantages of incorporating intercluster dependencies in the HDA clustering algorithm and equipping the algorithm with a general distortion measure that makes use of the HOS of the given data points. We show that, at the cost of moderate extra computation, it is possible to achieve good clustering performance even when the clusters overlap significantly.

ACKNOWLEDGMENT

The authors thank the anonymous reviewers for their insightful comments and suggestions.

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A New Approach to Synthesize Sharp 2-D Half-Band Filters

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Abstract—The frequency-response masking method is an efficient technique for the realization of sharp one-dimensional filters. Recently, this technique has been extended to the synthesis of sharp two-dimensional (2-D) filters. While it has been demonstrated that sharp diamond-shaped filters may be realized efficiently, the techniques previously presented cannot be applied to the synthesis of 2-D half-band filters. In this brief, we propose a modification of the technique for the synthesis of sharp 2-D half-band diamond-shaped filters. In addition, by exploiting the special properties of 2-D half-band filters, the complexities of the band-edge shaping and masking filters are further reduced. This results in a very efficient implementation of 2-D half-band diamond-shaped filters.

I. INTRODUCTION

Two fundamental operations in one-dimensional (1-D) digital signal processing (DSP) are decimation and interpolation by a factor of two. These operations introduce undesirable artifacts into the spectrum of the signal under consideration. To suppress these artifacts, anti-aliasing and anti-imaging filters are required. Finite-impulse response (FIR) filters having the half-band characteristic are natural choices in these applications. An FIR half-band filter has the important property that half of its coefficient values are trivial. Consequently, using a half-band filter in filtering operations yields a significant advantage in terms of computational complexity.

The conversion between signals sampled using the rectangular sampling lattice and the quincunx sampling lattice is an example of sampling rate alteration by a factor of two in the two-dimensional (2-D) case. In such a situation, the associated anti-aliasing and anti-imaging filter is the 2-D diamond-shaped (DS) filter. As in the 1-D case, the 2-D half-band DS filter is particularly attractive for such an application, since it has the desirable property that half of its coefficient values are trivial [4]. For this reason, we find that this filter is used in many video and image processing applications. For example, it can be used in the conversion between progressive and interlaced scanning motion pictures [5]–[7].

The frequency-response masking (FRM) technique [8], [9] is an efficient method for realizing sharp 1-D filters. In [1]–[3], it has been demonstrated that sharp 2-D DS filters may also be synthesized using the FRM technique. However, the partitioning scheme previously presented does not allow the synthesis of 2-D half-band DS filters. Furthermore, the realization of half-band filters would have required additional constraints on the sub-filters. In this paper, we introduce a new approach for the synthesis of sharp 2-D half-band DS filters and discuss the additional constraints required to realize half-band filters. In addition, we further reduce the complexity of the band-edge shaping and masking filters used in the FRM technique by taking advantage of the properties of half-band filters. A very efficient implementation structure of 2-D half-band filters is obtained.

The organization of this paper is as follows. In Section II, a description of the polyphase decomposition of 2-D filters is given. This Manuscript received May 8, 1998; revised April 16, 1999. This paper was recommended by Associate Editor M. Simaan.

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Publisher Item Identifier S 1057-7130(99)06542-8.
provides a framework for deriving the filter coefficient constraints on the sub-filters. Section III gives an overview of the FRM technique and how it may be applied to the synthesis of sharp 2-D half-band DS filters. To enforce the 2-D half-band characteristic, the band-edge shaping and masking filters must satisfy certain constraints; these constraints are discussed in Section IV. In Section V, we present how the various design parameters, such as the specifications of the band-edges of the sub-filters and the impulse response up-sampling ratio, may be selected. Finally, a design example is presented in Section VI to illustrate the effectiveness of our technique. Note that to simplify notation, we shall assume in this paper that all the filters are zero phase. Consequently, the resulting filters are noncausal. Nevertheless, causality can be easily achieved by delaying the impulse response of the filter by an appropriate number of samples.

II. POLYPHASE DECOMPOSITION

Consider a 2-D FIR linear phase filter $H(z_1, z_2)$ of support size $N \times N$, where $N$ is odd. Its transfer function can be written as

$$H(z_1, z_2) = \sum_{n_1, n_2} h(n_1, n_2)z_1^{-n_1}z_2^{-n_2}$$

(1)

where $h(n_1, n_2)$ represents the impulse response of the filter and the summation is taken for $n_1$ and $n_2$ ranging from $-\lceil(N-1)/2 \rceil$ to $\lceil(N-1)/2 \rceil$. The filter can be decomposed into four polyphase components according to

$$H(z_1, z_2) = [P_0 + P_1(z_1^2, z_2^2)] + z_1^{-1}z_2^{-1}P_2(z_1^2, z_2^2) + z_1^{-1}P_3(z_1^2, z_2^2) + z_2^{-1}P_1(z_1^2, z_2^2)$$

(2)

where $P_0, P_1(z_1, z_2), P_2(z_1, z_2), P_3(z_1, z_2)$ and $P_4(z_1, z_2)$ are the polyphase components. Note that we have intentionally sub-divided the first polyphase component into two parts, $P_0$ and $P_1(z_1^2, z_2^2)$, where $P_0 = h(0, 0)$. This is to facilitate the derivations of the conditions on the filter coefficients in Section IV.

Before proceeding further, it is useful to first define the 2-D half-band filter. $H_{b}(z_1, z_2)$ is said to be a 2-D half-band filter if

$$h(n_1, n_2) = \begin{cases} \frac{1}{2} & \text{if } n_1 = n_2 = 0 \\ 0, & \text{if } n_1 + n_2 \text{ is even and } (n_1 \neq 0, n_2 \neq 0). \end{cases}$$

(3)

From the definition, it is clear that approximately half of the filter’s coefficients are zero.

To give an insight on how the half-band filter may be obtained by using a series-cascade of filters, it is convenient to classify a 2-D filter according to the polyphase(s) the filter is composed of. For this purpose, we shall use the notations $Q_0, Q_1, Q_2, Q_3,$ and $Q_4$ to denote elementary filter classes and the angled-bracket pair $\langle \rangle$ to denote the classification operation. For example, if a filter $H_{b}(z_1, z_2)$ can be expressed as

$$H_{b}(z_1, z_2) = z_1^{-1}P_{b}(z_1^2, z_2^2)$$

(4)

then we classify $H_{b}(z_1, z_2)$ as class $Q_3$ and write

$$\langle H_{b}(z_1, z_2) \rangle = Q_3.$$ 

(5)

A 2-D filter may have more than one elementary class. For example, for the 2-D half-band filter $H_{b}(z_1, z_2)$, we have

$$\langle H_{b}(z_1, z_2) \rangle = Q_0 + Q_3 + Q_4$$

(6)

i.e., it has the composite class $Q_0 + Q_3 + Q_4$.

The rules corresponding to filters in series-cascade are as follows. If

$$\langle H_{b}(z_1, z_2) \rangle = Q_i$$

(7a)

$$\langle H_{b}(z_1, z_2) \rangle = Q_j$$

(7b)

then we have

$$\langle H_i(z_1, z_2)H_j(z_1, z_2) \rangle = Q_i \odot Q_j$$

(8)

where $\odot$ denotes the operation between $Q_i$ and $Q_j$ when $H_i(z_1, z_2)$ is cascaded to $H_j(z_1, z_2)$. The following axioms hold for the $\odot$ operation:

$$Q_i \odot Q_i = Q_i$$

(9a)

$$Q_i \odot Q_j = Q_j \odot Q_i$$

(9b)

$$Q_i \odot (Q_j + Q_k) = Q_i \odot Q_j + Q_i \odot Q_k.$$ 

(9c)

Furthermore, from the properties of the polyphasos defined in (2), it can be easily shown that the following also holds:

$$Q_1 \odot Q_3 = Q_3$$

(10a)

$$Q_1 \odot Q_4 = Q_1$$

(10b)

$$Q_2 \odot Q_3 = Q_4$$

(10c)

$$Q_2 \odot Q_4 = Q_3.$$ 

(10d)

These results are used in Section IV to illustrate how the 2-D half-band filter may be synthesized.

III. FREQUENCY-RESPONSE MASKING

The FRM technique for the synthesis of sharp 1-D filters has been investigated in detail in the literature [8], [9]. The extension of the technique to 2-D was first introduced in [1], and subsequently generalized in [2]. Basically, the technique is based on the concept that up-sampling the impulse response of a filter by inserting zeros reduces its transition width; if the up-sampling ratio is $M$, then the transition width will be reduced to one-$M$th of the original value.

Two types of filters are used in the FRM technique—the band-edge shaping filters $F_{b}(z_1, z_2)$, $F_{b}(z_1, z_2)$, $F_{c}(z_1, z_2)$, and $F_{d}(z_1, z_2)$, and the masking filters $F_{MA}(z_1, z_2)$, $F_{MB}(z_1, z_2)$, $F_{MC}(z_1, z_2)$, and $F_{MD}(z_1, z_2)$. The frequency responses of the ideal band-edge shaping filters are illustrated in Fig. 1(a); they have a gain of unity in the indicated region and zero elsewhere.

The block diagram of the 2-D FRM technique is shown in Fig. 2. In Fig. 2, $F_{b}(z_1^M, z_2^M)$, $F_{b}(z_1^M, z_2^M)$, $F_{c}(z_1^M, z_2^M)$ and $F_{d}(z_1^M, z_2^M)$ are obtained by up-sampling the impulse responses of the band-edge shaping filters by a pre-determined ratio $M$. Their frequency responses for the case where $M = 3$ are shown in Fig 1(b). These filters partition the frequency spectrum into the complementary components, $A(z_1, z_2)$, $B(z_1, z_2)$, $C(z_1, z_2)$, and $D(z_1, z_2)$. The complementary components are processed and shaped by masking filters that attenuate undesired frequency portions. The frequency responses of the masking filters $F_{MA}(z_1, z_2)$, $F_{MB}(z_1, z_2)$, $F_{MC}(z_1, z_2)$, and $F_{MD}(z_1, z_2)$ are that illustrated in Fig. 1(c)-(f), respectively. In Fig. 1(c)-(f), the passbands and stopbands of the masking filters are shown shaded.

The final output is obtained from the summation of the “processed” complementary components. Mathematically, the equivalent filter $F(z_1, z_2)$ is given by

$$F(z_1, z_2) = F_{a}(z_1^M, z_2^M)F_{MA}(z_1, z_2) + F_{b}(z_1^M, z_2^M)F_{MB}(z_1, z_2) + F_{c}(z_1^M, z_2^M)F_{MC}(z_1, z_2) + F_{d}(z_1^M, z_2^M)F_{MD}(z_1, z_2).$$ 

(11)

Using this technique, $F(z_1, z_2)$ can implement a filter with a very narrow transition width with significant savings in computational complexity. The savings stem from the fact that the band-edge shaping filters have sparse coefficients and that the masking filters have wide transition bands when compared to the filter $F(z_1, z_2)$. 
The 2-D half-band filter that is obtained for the case $M = 3$ is illustrated in Fig. 1(b).

From Fig. 1, it can be noted that the regions of support for all the band-edge shaping filters have the same diamond shape, except that they are translated from one another in the frequency domain. This suggests that the four band-edge shaping filters can be derived from a single filter. We shall call this filter the prototype band-edge shaping filter $F_{\text{un}}(z_1, z_2)$. Similarly, we can apply the same idea to the masking filters; we define a prototype masking filter $F_{m}(z_1, z_2)$, from which the four masking filters can be derived. The frequency responses of the two prototype filters are illustrated graphically in Fig. 3. Note that the definitions of the band-edges for $F_{\text{un}}(z_1, z_2)$ and $F_{m}(z_1, z_2)$ are different; for $F_{\text{un}}(z_1, z_2)$, we have $\omega + \omega = \pi$, and for $F_{m}(z_1, z_2)$, we have $\omega + \omega = 2\pi$. Both the prototype filters have sparse coefficients. For $F_{\text{un}}(z_1, z_2)$, we have $f_{\text{un}}(n_1, n_2) = 0$ when $n_1 + n_2$ is odd—a result that can be obtained by considering the symmetry of the frequency response about $(\pi/2, \pi/2)$. $F_{\text{un}}(z_1, z_2)$ is a half-band filter and thus $f_{m}(n_1, n_2) = 0$ when $n_1 + n_2$ is even (with the exception of $f_{m}(0, 0)$, which has a value of 0.5).

The filters $F_A(z_1, z_2)$, $F_B(z_1, z_2)$, $F_C(z_1, z_2)$, and $F_D(z_1, z_2)$ are obtained by translating the frequency response of $F_{\text{un}}(z_1, z_2)$ by $(\pi/2, 0)$, $(-\pi/2, 0)$, $(0, -\pi/2)$, and $(0, \pi/2)$, respectively, in the frequency domain as follows. Let

$$F_{\text{un}}(z_1, z_2) = \sum_{n_1, n_2} f_{\text{un}}(n_1, n_2) z_1^{-n_1} z_2^{-n_2} \tag{12}$$

where the summation is taken for $n_1$ and $n_2$ ranging from $-(N_{\text{un}} - 1)/2$ to $(N_{\text{un}} - 1)/2$, then by replacing $z_1$ by $z_1 e^{(j\pi)/2}$, we obtain

$$F_A(z_1, z_2) = \sum_{n_1, n_2} f_{\text{un}}(n_1, n_2) e^{(j\pi)/2} z_1^{-n_1} z_2^{-n_2} \tag{13}$$

and we have

$$F_A(z_1^M, z_2^M) = \sum_{n_1, n_2} f_{\text{un}}(n_1, n_2) e^{(j\pi)/2} z_1^{-n_1 M} z_2^{-n_2 M} \tag{14}$$

The term $e^{(j\pi)/2}$ in (14) is either real or purely imaginary. With $f_{\text{un}}(n_1, n_2)$ being real, each term on the right-hand side of (14) is therefore either real or purely imaginary. If we write $F_A(z_1^M, z_2^M)$ in terms of the real and imaginary parts

$$F_A(z_1^M, z_2^M) = F_{A, R}(z_1^M, z_2^M) + j F_{A, I}(z_1^M, z_2^M) \tag{15}$$

where the subscripts $R$ and $I$ denote the real and imaginary parts of the filter, then the number of nonzero coefficients in $F_A(z_1^M, z_2^M)$ is given by the summation of the number of nonzero coefficients in $F_{A, R}(z_1^M, z_2^M)$ and $F_{A, I}(z_1^M, z_2^M)$.

For the $B(z_1, z_2)$ component, $F_B(z_1^M, z_2^M)$ is obtained by first replacing $z_1$ by $z_1 e^{(j\pi)/2}$ in (12) and then up-sampling the impulse response. It is given by

$$F_B(z_1^M, z_2^M) = \sum_{n_1, n_2} f_{\text{un}}(n_1, n_2) e^{(j\pi)/2} z_1^{-n_1 M} z_2^{-n_2 M}. \tag{16}$$
The $C(z_1, z_2)$ and $D(z_1, z_2)$ components are obtained similarly by replacing $z_2$ by $z_2 e^{j \pi/2}$ and $z_2$ by $z_2 e^{-j \pi/2}$, respectively, in (12) and then up-sampling the impulse responses.

The masking filters $F_{MA}(z_1, z_2)$, $F_{MB}(z_1, z_2)$, $F_{MC}(z_1, z_2)$, and $F_{MD}(z_1, z_2)$ are obtained by translating the frequency response of the prototype masking filter $F_m(z_1, z_2)$ in the frequency domain by $(\vartheta, 0)$, $(0, \vartheta)$, $(0, 0)$, and $(0, 0)$, respectively, for $M$ odd, and by $(\vartheta, 0)$, $(0, \vartheta)$, $(0, 0)$, and $(0, 0)$, respectively, for $M$ even. For both cases, the amount of translation $\vartheta$ is given by

$$\vartheta = \frac{\pi}{2M} = \pi \Delta_h$$  \hspace{1cm} (17)

where $\Delta_h$ is the normalized transition width of the desired 2-D half-band filter. The expressions for $F_{MA}(z_1, z_2)$ and $F_{MB}(z_1, z_2)$ for the case where $M$ is odd are

$$F_{MA}(z_1, z_2) = \sum_{n_1, n_2} f_m(n_1, n_2) e^{j n_1 \vartheta} z_2^{-n_1} z_2^{-n_2}$$

$$F_{MB}(z_1, z_2) = \sum_{n_1, n_2} f_m(n_1, n_2) e^{-j n_1 \vartheta} z_2^{-n_1} z_2^{-n_2}$$  \hspace{1cm} (18)

In (18), the values of $n_1$ and $n_2$ range from $-(N_m - 1)/2$ to $(N_m - 1)/2$.

From (14), (16), and (18), it is clear that the coefficients of $F_B(z_1^{M}, z_2^{M})$ are the complex conjugate of those from $F_A(z_1^{M}, z_2^{M})$ and the coefficients of $F_{MB}(z_1, z_2)$ are the complex conjugate of those from $F_{MA}(z_1, z_2)$. This also applies for the filters for the $C(z_1, z_2)$ and $D(z_1, z_2)$ components, regardless of whether $M$ is odd or even. Thus, after some manipulation, it can be shown that expression (11) reduces to

$$F(z_1, z_2) = 2 \left\{ \text{Re} \left[ F_A(z_1^{M}, z_2^{M}) F_{MA}(z_1, z_2) \right] + \text{Re} \left[ F_C(z_1^{M}, z_2^{M}) F_{MC}(z_1, z_2) \right] \right\}$$  \hspace{1cm} (19)

where $\text{Re}(z)$ denotes the real part of $z$. The block diagram that implements the above expression is illustrated in Fig. 4. From the given sub-filters' characteristics, it can be further shown using (19) that $F(z_1, z_2)$ has an eight-fold symmetry in the frequency domain.

**Fig. 3.** (a) Prototype band-edge shaping filter. (b) Prototype masking filter.

**IV. CONDITIONS ON THE FILTER COEFFICIENTS**

While a 2-D DS filter synthesized using the structure shown in Fig. 4 is guaranteed to have an eight-fold symmetry, it is not sufficient to ensure that the filter will be a half-band filter. To ensure that the synthesized filter has the half-band characteristic, additional constraints must be imposed on the band-edge shaping and masking filters. We shall now examine these constraints.

Consider the transfer function of the synthesized filter $F(z_1, z_2)$

$$F(z_1, z_2) = 2 \left[ F_{A,R}(z_1^{M}, z_2^{M}) F_{MA,R}(z_1, z_2) \right.$$  $$+ F_{C,R}(z_1^{M}, z_2^{M}) F_{MC,R}(z_1, z_2)$$

$$- F_{A,I}(z_1^{M}, z_2^{M}) F_{MA,I}(z_1, z_2)$$

$$- F_{C,I}(z_1^{M}, z_2^{M}) F_{MC,I}(z_1, z_2) \right].$$  \hspace{1cm} (20)

Using the concept of filter class introduced in Section II, we have

$$\langle F_{A,R}(z_1^{M}, z_2^{M}) F_{MA,R}(z_1, z_2) \rangle = Q_0 + Q_1$$  \hspace{1cm} (21a)

$$\langle F_{A,I}(z_1^{M}, z_2^{M}) F_{MA,I}(z_1, z_2) \rangle = Q_2$$  \hspace{1cm} (21b)

$$\langle F_{MA,R}(z_1, z_2) \rangle = Q_0 + Q_3 + Q_4$$  \hspace{1cm} (21c)

$$\langle F_{MA,I}(z_1, z_2) \rangle = Q_3 + Q_4.$$  \hspace{1cm} (21d)

Applying the results in (10), we have

$$\langle F_{A,R}(z_1^{M}, z_2^{M}) F_{MA,I}(z_1, z_2) \rangle = \langle F_{C,R}(z_1^{M}, z_2^{M}) F_{MC,I}(z_1, z_2) \rangle = Q_3 + Q_4.$$  \hspace{1cm} (22)

and

$$\langle F_{A,I}(z_1^{M}, z_2^{M}) F_{MA,R}(z_1, z_2) \rangle$$

$$= \langle F_{C,I}(z_1^{M}, z_2^{M}) F_{MC,R}(z_1, z_2) \rangle$$

$$= (Q_0 + Q_1) \ominus (Q_0 + Q_3 + Q_4)$$

$$= [(Q_0 + Q_1) \ominus Q_0] + [(Q_0 + Q_1) \ominus (Q_3 + Q_4)]$$

$$= [Q_0 + Q_1] + [Q_3 + Q_4].$$  \hspace{1cm} (23)

Recall from (6) that a 2-D half-band filter has the class $Q_0 + Q_3 + Q_4$. Thus, for $F(z_1, z_2)$ to be a half-band filter, the filter coefficients that contribute to the $Q_1$ term in (23) must vanish. Furthermore, if $f(n_1, n_2)$ represents the impulse response of the filter $F(z_1, z_2)$, $f(0, 0)$ should have a value of 0.5.

Consider the term $[(Q_0 + Q_1) \ominus Q_0]$ in (23). From (18), the $Q_0$ term from $F_{MA,R}(z_1, z_2)$ and $F_{MC,R}(z_1, z_2)$ has a value given by

$$\text{Re}[f_{m,0}(0, 0)] = \text{Re}[f_{m,0}(0, 0)] = f_m(0, 0) = 0.5.$$  \hspace{1cm} (24)

Thus, such $Q_0$ term is just a scaling factor for the $(Q_0 + Q_1)$ term from $F_{A,R}(z_1^{M}, z_2^{M})$ and $F_{C,R}(z_1^{M}, z_2^{M})$ in (23). To ensure that there are no $Q_1$ term in $[(Q_0 + Q_1) \ominus Q_0]$, it is only necessary to check that no $Q_1$ term appears in the summation of the corresponding terms from
the coefficients of the filters $F_{A,H}(z^{1/2}, z^{3/2})$ and $F_{C,R}(z^{1/2}, z^{3/2})$. Let $f_{min}(n_1, n_2)$ be this summation, then
\[
f_{min}(n_1, n_2) = f_{hu}(n_1, n_2) \cos \frac{n_1 \pi}{2} + f_{hn}(n_1, n_2) \cos \frac{n_2 \pi}{2} = 2 f_{hu}(n_1, n_2) \left[ \cos \left( \frac{n_1 + n_2}{4} \pi \right) \cos \left( \frac{n_1 - n_2}{4} \pi \right) + \frac{n_1 + n_2}{4} \right].
\] (25)

When $n_1$ and $n_2$ are both zero, we require
\[
2 f_{hu}(n_1, n_2) f_m(0, 0) = 4 f_{hu}(0, 0) f_m(0, 0) = 0.5.
\] (26)

So $f_{hu}(0, 0) = 0.25$. For all other values of $n_1$ and $n_2$, $f_{min}(n_1, n_2)$ should be zero. To enforce this, $f_{hu}(n_1, n_2)$ must be set to zero when the term $\cos ((n_1 + n_2)\pi/2) \cos ((n_1 - n_2)\pi/2)$ is nonzero. Thus, the overall condition for $f_{hu}(n_1, n_2)$ is
\[
f_{hu}(n_1, n_2) = \begin{cases} 0.25, & \text{if } n_1 = n_2 = 0, \\ 0, & \text{if } n_1 + n_2 = 4 \kappa_1, \quad n_1 - n_2 = 4 \kappa_2 \\ & \text{(except } n_1 = n_2 = 0) \end{cases}
\] (27)

where $\kappa_1$ and $\kappa_2$ are integers.

With the above condition, the filter $F(z_1, z_2)$ obtained would be a 2-D half-band filter. One interesting observation is that, as shown in the Appendix, the condition also ensures that $F_{A}(z^{1/2}, z^{3/2}), F_{B}(z^{1/2}, z^{3/2}), F_{C}(z^{1/2}, z^{3/2}),$ and $F_{D}(z^{1/2}, z^{3/2})$ form a set of complementary filters. This means that the usage of the error component mentioned in [1], [2] is not required.

V. Design Parameters

A. Band-Edge Specifications

Let $\omega_p$ and $\omega_s$ denote the frequency values where the passband- and stopband-edges of the prototype band-edge shaping and masking filter meet the frequency axes, as shown in Fig. 3. Also let $\Delta$ denote the normalized transition width, i.e., $\Delta = (\omega_s - \omega_p)/(2\pi)$.

Suppose the passband- and stopband-edges of the desired 2-D half-band filter $F(z_1, z_2)$ are $\psi_p$ and $\psi_s$, respectively, and that its transition width is $\Delta_h = (\psi_s - \psi_p)/(2\pi)$. If $M$ is the selected impulse response up-sampling ratio, then the prototype band-edge shaping filter has a transition width given by $\Delta_{h, \omega} = M \Delta_h$, and its passband- and stopband-edges are given by $\omega_p = (\pi - \Delta_{h, \omega})/2$ and $\omega_s = (\pi + \Delta_{h, \omega})/2$, respectively. For the prototype masking filter, its transition width is given by $\Delta_{h, \psi} = 1/(2M)$ and its band-edges are located at $\omega_p = \pi (1 - \Delta_{h, \psi})$ and $\omega_s = \pi (1 + \Delta_{h, \psi})$.

B. Ripple Magnitude

Upper bounds for the ripple magnitudes of the synthesized 2-D half-band filter in the passband and stopband can be obtained by considering the ripple contributions from each of the sub-filters. The derivation is lengthy but straightforward and is very similar to that presented in [3]. Thus, we shall not present the derivation here. A summary of the final results is as follows. If $\delta_{h, \omega}$ and $\delta_m$ are the peak ripple magnitudes of the prototype band-edge shaping filter and the prototype masking filter, respectively, then an upper bound for the peak ripple magnitude $\delta_h$ of the synthesized 2-D half-band filter is given by
\[
\delta_h \leq 2 \delta_{h, \omega} + \delta_m.
\] (28)

This equation applies in both the passband and stopband.

C. Up-Sampling Ratio $M$

We shall now derive an expression for the value of $M$ that minimizes the FRM implementation complexity. The complexity of a particular implementation shall be assumed to be directly proportional to the number of multipliers required in the implementation. To facilitate the derivations, we need to know the relationships between the passband and stopband ripple magnitudes $\delta_p$ and $\delta_s$, filter support size $N$ by $N_s$ and transition width $\Delta$ of the filters used in the synthesis. The design rules in [10] can be utilized for this purpose. The design rules state that for a rectangular-shaped filter, the relationship between the above variables can be approximated by
\[
N \approx \frac{\Phi(\delta_p, \delta_s)}{\Delta} + 1
\] (29)
where
\[
\Phi(\delta_p, \delta_s) = -0.299 - 0.568 \log_{10} \delta_p - 0.852 \log_{10} \delta_s.
\] (30)

Since $\Delta$ is small, the term $(\Phi(\delta_p, \delta_s))/\Delta$ is dominant, and we shall write
\[
N \approx (\Phi(\delta_p, \delta_s))/\Delta.
\] (31)

Consider a 2-D DS half-band filter $F(z_1, z_2)$ with passband and stopband ripple magnitudes of no more than $\delta_{h, \omega}$, normalized transition width of $\Delta_h$, and a support size of $N_h$ by $N_s$. Using (31), the number of multipliers $P_h$ required to implement this filter is given by
\[
P_h = \frac{N_h^2}{2} \approx \frac{3}{4} \Phi(\delta_{h, \omega}, \delta_{h, \omega})^2.
\] (32)

Note that the factor of half in (32) arises because of the filter’s half-band characteristic.

Suppose that the FRM technique is used to synthesize this filter. Let the filter support sizes of the prototype band-edge shaping and prototype masking filter be given by $N_{h, \omega}$ by $N_{h, \omega}$ and $N_{m}$ by $N_{m}$, respectively. The transition width of the prototype band-edge shaping filter is $M$ times the transition width of $F(z_1, z_2)$. So, we have
\[
N_{h, \omega} = \frac{\Phi(\delta_{h, \omega}, \delta_{h, \omega})}{M \Delta_h}.
\] (33)

The coefficients of the prototype band-edge shaping filter satisfy the conditions stated in Section IV. It can be shown that for such a filter with a support size of $N_{h, \omega}$ by $N_{h, \omega}$, the number of nonzero coefficients is given by $(3N_{h, \omega}^2)/8$. Thus, the total complexity of the filters $F_{A,R}(z^{1/2}, z^{3/2}), F_{A,I}(z^{1/2}, z^{3/2}), F_{C,R}(z^{1/2}, z^{3/2}),$ and $F_{C,I}(z^{1/2}, z^{3/2})$ is given by
\[
P_{h, m} = \frac{3}{4} \Phi(\delta_{h, \omega}, \delta_{h, \omega})^2.
\] (34)

The transition width of the prototype masking filter is given by $1/(2M)$. Thus we have
\[
N_m = \frac{\Phi(\delta_{h, \omega}, \delta_{h, \omega})}{1/(2M)} = 2M \Phi(\delta_{h, \omega}, \delta_{h, \omega}).
\] (35)

Since the prototype masking filter is a half-band filter, the total complexity of the filters $F_{MA,H}(z_1, z_2), F_{MA,I}(z_1, z_2), F_{M,R}(z_1, z_2),$ and $F_{M,I}(z_1, z_2)$ is given by
\[
P_m = \frac{3}{4} N_m^2 = 4 M^2 \Phi(\delta_m, \delta_m).\] (36)

Finally, the complexity of the implementation $P_t$ is as follows:
\[
P_t = P_{h, m} + P_m = \frac{3}{4} \Phi(\delta_{h, \omega}, \delta_{h, \omega})^2 + 4 M^2 \Phi(\delta_m, \delta_m).
\] (37)
From (37), it is clear that when the value of $M$ is increased, the complexity of the band-edge shaping filters decreases while that for the masking filters increases and vice versa. Therefore, an optimum value of $M$ that minimizes the total complexity $P_{tl}$ can be obtained. By differentiating $P_{tl}$ with respect to $M$ and equating the derivative to zero, it can be shown that $P_{tl}$ is minimized when $P_{t0} = P_{m}$. The value of $M$ at minimum complexity $M_{opt}$ is given by

$$M_{opt} = \frac{\sqrt{\frac{24}{\Delta b}}}{\Phi_2(\delta_{h0}, \delta_{h})} \left[ \frac{\Delta h \Phi_2(\delta_{m}, \delta_{m})}{\Delta b} \right]^{1/2}.$$  

(38)

The above expression provides valuable information on how the value of $M$ can be selected. If the prototype filters are designed such that $\delta_{h0} = \delta_{m}$, then $M_{opt}$ is simply given by

$$M_{opt} = \frac{\sqrt{\frac{24}{\Delta b}}}{\Phi_2(\delta_{h0}, \delta_{h})} \approx 0.5533 \sqrt{\Delta h}.$$  

(39)

and the minimum complexity is

$$P_{mn} = \frac{\sqrt{64\Delta b^2(\delta_{h}, \delta_{m})}}{\Delta b} \approx \frac{\sqrt{64\Delta b^2(\delta_{h}, \delta_{m})}}{\Delta b}.$$  

(40)

where $\delta_{h} = \delta_{h0} = \delta_{m}$.

VI. DESIGN EXAMPLE

We illustrate the synthesis of a 2-D half-band filter with a transition width of $0.02\pi$ and with ripple magnitudes of not more than $0.05$ ($-26.0$ dB) in both the passbands and stopbands. We shall design the prototype band-edge shaping filter and the prototype masking filter, such that $\delta_{h0} = \delta_{m}$. From (39), the value of $M$ at minimum complexity is given by $M_{opt} \approx 6$. With this value of $M$, the transition widths of the prototype band-edge shaping filter and prototype masking filter are given by 0.06 and 0.0833, respectively. The specifications of the band-edges of these filters can be obtained from Section V.A.

From (28), the prototype band-edge shaping filter and prototype masking filter should be designed to have ripple magnitudes of not more than 0.0167 ($-35.6$ dB) in both the passbands and stopbands. These filters can be designed using the linear programming technique and filters with $N_{bo} = 39$ and $N_{m} = 33$ satisfy this requirement. With these sub-filters, we are able to synthesize the 2-D half-band filter with the frequency response magnitude plot shown in Fig. 5(a).

Its frequency response along the diagonal $\omega_1 = \omega_2$ is shown in Fig. 5(b). Using a very dense frequency grid (4096 by 4096 points in the region $[-\pi, \pi]^2$) to check the frequency response of the synthesized filter, the passband and stopband ripple magnitudes were found to be not more than $-28.0$ dB and, thus, satisfies the design requirements.

A filter that meets this set of frequency response specifications has a minimax support size of 201 by 201 [10]–[12]. Its complexity in terms of number of multipliers per output sample would be 40000. With our technique, this number reduces to approximately $2((3N_{bo}^2)/8) + 4\left(\frac{N_{m}^2}{2}\right) \approx 33000$. There is a saving in terms of the computational complexity by an order of magnitude. A further advantage of technique worth mentioning is that the impulse response of the synthesized filter (which in this case is 261 by 261) is only slightly larger than the minimax minimum. This is considerably better than various separable implementations of DS filters.

VII. CONCLUSION

A new approach for the synthesis of sharp 2-D FIR diamond-shaped half-band filters using the frequency response masking technique is presented. The method uses the special properties of the 2-D half-band filter to reduce the complexities of the band-edge shaping and masking filters. The conditions that must be imposed on the sub-filters in order to obtain the half-band characteristic are presented. An expression for the impulse response up-sampling ratio that produces the design with the minimum complexity is derived and the effectiveness of the technique is illustrated with a design example. When the transition width of the desired 2-D half-band filter is narrow, a very significant amount of complexity reduction can be achieved.

APPENDIX

From Section III, the $(n_1, n_2)$th coefficient of the filters $F_A(\lambda_1, \lambda_2)$, $F_B(\lambda_1, \lambda_2)$, $F_C(\lambda_1, \lambda_2)$, and $F_D(\lambda_1, \lambda_2)$ are given by

$$f_{bo}(n_1, n_2)e^{j(n_1\pi/2)}f_{bo}(n_1, n_2)e^{j(n_2\pi/2)}, \quad f_{bo}(n_1, n_2)e^{-j(n_1\pi/2)}, \quad f_{bo}(n_1, n_2)e^{-j(n_2\pi/2)}$$

and

$$f_{bo}(n_1, n_2)e^{j(n_1\pi/2)}f_{bo}(n_1, n_2)e^{-j(n_2\pi/2)}.$$  

respectively. The summation of the four terms is

$$f_{bo}(n_1, n_2)\left[e^{j(n_1\pi/2)} + e^{-j(n_1\pi/2)} + e^{j(n_2\pi/2)} + e^{-j(n_2\pi/2)}\right]$$

$$= 4f_{bo}(n_1, n_2)\cos((n_1 + n_2)\pi/4)\cos\left(\frac{(n_1 - n_2)\pi}{4}\right).$$  

(41)
With the condition listed in (27), this summation is unity for \( n_1 = n_2 = 0 \) and zero for all other values of \( n_1 \) and \( n_2 \), i.e., the four band-edge shaping filters form a complementary set.

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Non-Wiener Behavior of the Filtered LMS Algorithm

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Abstract—This brief presents some analytical results for the non-Wiener solutions of the filtered least mean square algorithm. Secondary paths are modeled as linear time-invariant filters. Results are presented for the steady-state adaptive weight behavior and algorithm stability. Stability regions are implicitly determined as a function of the algorithm step size \( \mu \), the number of filter taps \( N \), and the reference signal power.

Index Terms—Acoustic noise, adaptive control, adaptive filters, adaptive signal processing, interference suppression, least mean square methods, notch filters, stability.

I. INTRODUCTION

The filtered least mean square (FLMS) adaptive algorithm is often used in active noise control (ANC) applications [1], [2]. Ducts, speakers, microphones, and/or other acoustic devices filter the signals in the ANC adaptation path. These secondary-path effects [1] are usually modeled as linear time-invariant (LTI) filters using a transfer function description.

The behavior of the FLMS algorithm has been studied by several authors [1]–[6]. LMS adaptation was studied for a filter located at the adaptive filter output in [3]. The delayed coefficient LMS algorithm (a particular case of FLMS) has been studied for stochastic inputs in [4], [5]. Algorithm stability using a frequency-domain approach was studied in [6]. These papers studied FLMS stability for broad-band stochastic reference inputs. Often, ANC applications involve the cancellation of narrow-band sinusoidal reference signals [1], [7], [8]. Narrow-band ANC is important for certain industrial applications such as the cancellation of periodic noise from engines, compressors, motors, fans, and propellers. It is well known that the LMS algorithm converges to a notch filter for narrow-band references [9]–[11]. A deterministic set of recursions describes its behavior and yields non-Wiener solutions [10], [11]. Non-Wiener behavior occurs, for example, when the primary and reference signals are uncorrelated, yet the steady-state mean weight vector is nonzero. In these cases, algorithm behavior cannot be predicted using results obtained from the Wiener theory. This non-Wiener behavior has been studied in [9]–[14]. However, these analyses are not valid when there are filters in the adaptation path. The adaptive behavior depends upon the filter parameters. For example, the stability region (as a function of the step size \( \mu \)) has been shown to be smaller than predicted by existing theory for the LMS algorithm [15], [16].

This brief presents some results for the non-Wiener behavior of the FLMS algorithm using the time domain approach in [12]. The primary and reference inputs are noiseless sinusoids. The non-Wiener algorithm behavior is determined for primary and reference signals at the same and different frequencies. The weight behavior is shown to be quite different for the two cases. Also, the present analysis allows for three distinct transfer functions within the cancellation

Manuscript received July 15, 1998; revised May 11, 1999. This work was supported in part by the Brazilian National Council for Development of Science and Technology (CNPq) under Grant 352084/92-8. This paper was recommended by Associate Editor N. R. Shanbhag.

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Publisher Item Identifier S 1057-7130(99)06525-8.